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An extreme point characterization of strategy-proof and unanimous probabilistic rules over binary restricted domains[☆]

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ABSTRACT

We show that every strategy-proof and unanimous probabilistic rule on a binary restricted domain has binary support, and is a probabilistic mixture of strategy-proof and unanimous deterministic rules. Examples of binary restricted domains are single-dipped domains, which are of interest when considering the location of public bads. We also provide an extension to infinitely many alternatives.

1. Introduction

Suppose that in choosing between red and white wine, half of the dinner party is in favor of red wine while the other half prefers white wine. In this situation a deterministic (social choice) rule has to choose one of the two alternatives, where a fifty–fifty lottery seems to be more fair. In general, for every preference profile a probabilistic rule selects a lottery over the set of alternatives. Gibbard (1977) provides a characterization of all strategy-proof probabilistic rules over the complete domain of preferences (see also Sen, 2011). In particular, if in addition a rule is unanimous, then it is a probabilistic mixture of deterministic rules. This result implies that in order to analyze probabilistic rules it is sufficient to study deterministic rules only.

In Peters et al. (2014) it is shown that if preferences are single-peaked over a finite set of alternatives then every strategy-proof and unanimous probabilistic rule is a mixture of strategy-proof and unanimous deterministic rules.[1] The same is true for the multi-dimensional domain with lexicographic preferences (Chatterji et al., 2012). But it is not necessarily true for all dictatorial domains (Chatterji et al., 2014), in particular, there are domains where all strategy-proof and unanimous deterministic rules are dictatorial but not all strategy-proof and unanimous probabilistic rules are random dictatorships.

A binary restricted domain over two alternatives x and y is a domain of preferences where the top alternative(s) of each preference belong(s) to the set $\{x, y\}$ (we allow for indifferences); and moreover, for every preference with top x there is a preference with top y such that the only alternatives weakly preferred to y in the former and x in the latter preference, are x and y.

Outstanding examples of binary restricted domains are domains of single-dipped preferences with respect to a given ordering of the alternatives. Single-dipped preferences are of central interest in situations where the location of an obnoxious facility (public bad) has to be determined by voting: think of deciding on the location of a garbage dump along a road, such that every inhabitant has a single dip (his house, or the school of his children, etc.) and prefers a location for the garbage dump as far away as possible from this dip. Peremans and Storcken (1999) have shown the equivalence between individual and group strategy-proofness in subdomains of single-dipped preferences. They characterize a general class of strategy-proof deterministic rules. In Manjunath (2014) the problem of locating a single public bad along a line segment when agents' preferences are single-dipped, is studied. In particular, all strategy-proof and unanimous deterministic rules are characterized. In Barberà et al. (2012) it is shown that, when all single-dipped preferences are admissible, the range

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1 Ehlers et al. (2002) characterize such probabilistic rules for single-peaked preferences where the set of alternatives is the real line.

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of a strategy-proof and unanimous deterministic rule contains at most two alternatives. The paper also provides examples of subdomains admitting strategy-proof rules with larger ranges. Ayllón and Caramuta (2016) consider group strategy-proofness under single-dipped preferences when agents become satiated: above a certain distance from their dips they become indifferent, and thus they go beyond the binary restricted domain. Further works on strategy-proofness under single-dipped preferences include Öztürk et al. (2013, 2014), Lahiri et al. (forthcoming), and Chatterjee et al. (2016). For strong Nash implementation under single-dipped preferences see Yamamura (2016). There is also a literature on this topic when side payments are allowed, e.g., Lescop (2007) or Sakai (2012).

In the present paper we show that every strategy-proof and unanimous probabilistic rule over a binary restricted domain with top alternatives x and y has binary support, i.e., for every preference profile probability 1 is assigned to $\{x, y\}$. We also show that if a strategy-proof and unanimous probabilistic rule has binary support then it can be written as a convex combination of deterministic rules. Moreover, we present a complete characterization of such rules, by using so-called admissible collections of committees.

Closely related papers are Larsson and Svensson (2006) and Picot and Sen (2012). Larsson and Svensson (2006) include a characterization of all strategy-proof surjective deterministic rules for the case of two alternatives with indifferences allowed. Their Theorem 3 is close to our Theorem 3.9—our theorem is slightly more general since we allow for more than two alternatives. Picot and Sen (2012) show that every probabilistic rule is a convex combination of deterministic rules if there are only two alternatives and no indifferences are allowed.

The paper is organized as follows. Section 2 introduces the model and definitions. Section 3 contains the main results, Section 4 contains an application to single-dipped preference domains, and Section 5 presents an extension to the case where the number of alternatives may be infinite.

2. Preliminaries

Let A be a finite set of at least two alternatives and let $N = \{1, \ldots, n\}$ be a finite set of at least two agents. Subsets of N are called *coalitions*. Let $\mathbb{W}(A)$ be the set of (*weak*) *preferences* over A.[2] By P and I we denote the asymmetric and symmetric parts of $R \in \mathbb{W}(A)$. For $R \in \mathbb{W}(A)$ by $\tau(R)$ we denote set of the first ranked alternative(s) in R, i.e., $\tau(R) = \{x \in A : xRy \text{ for all } y \in A\}$. In general, the notation $\mathcal{D}$ will be used for a set of admissible preferences for an agent $i \in N$. As is clear from the notation, we assume the same set of admissible preferences for every agent. A *preference profile*, denoted by $R_N = (R_1, \ldots, R_n)$, is an element of $\mathcal{D}^n$, the Cartesian product of n copies of $\mathcal{D}$. For a coalition S, R_S denotes the restriction of R_N to S. For notational convenience we often denote a singleton set $\{z\}$ by z.

Definition 2.1. A *deterministic rule* (DR) is a function $f : \mathcal{D}^n \to A$.

Definition 2.2. A DR f is *unanimous* if $f(R_N) \in \cap_{i=1}^n \tau(R_i)$ for all $R_N \in \mathcal{D}^n$ such that $\cap_{i=1}^n \tau(R_i) \neq \emptyset$.

Agent $i \in N$ *manipulates* DR f *at* $R_N \in \mathcal{D}^n$ *via* R_i' if $f(R_i', R_{N\setminus i}) P_i f(R_N)$.

Definition 2.3. A DR f is *strategy-proof* if for all $i \in N$, $R_N \in \mathcal{D}^n$, and $R_i' \in \mathcal{D}$, i does not manipulate f at R_N via R_i'.

[2] I.e., for all $R \in \mathbb{W}(A)$ and $x, y, z \in A$, we have xRy or yRx (completeness), and xRy and yRz imply xRz (transitivity). Note that reflexivity (xRx for all $x \in A$) is implied.

Definition 2.4. A *probabilistic rule* (PR) is a function $\Phi : \mathcal{D}^n \to \triangle A$ where $\triangle A$ is the set of probability distributions over A. A *strict* PR is a PR that is not a DR.

Observe that a deterministic rule can be identified with a probabilistic rule by assigning probability 1 to the chosen alternative.

For $a \in A$ and $R_N \in \mathcal{D}^n$, $\Phi_a(R_N)$ denotes the probability assigned to a by $\Phi(R_N)$. For $B \subseteq A$, we denote $\Phi_B(R_N) = \sum_{a\in B} \Phi_a(R_N)$.

Definition 2.5. A PR Φ is *unanimous* if $\Phi_{\cap_{i=1}^n \tau(R_i)}(R_N) = 1$ for all $R_N \in \mathcal{D}^n$ such that $\cap_{i=1}^n \tau(R_i) \neq \emptyset$.

Definition 2.6. For $R \in \mathcal{D}$ and $x \in A$, the *upper contour set* of x at R is the set $U(x, R) = \{y \in X : yRx\}$. In particular, $x \in U(x, R)$.

Agent $i \in N$ *manipulates* PR Φ *at* $R_N \in \mathcal{D}^n$ *via* R_i' if $\Phi_{U(x,R_i)}(R_i', R_{N\setminus i}) > \Phi_{U(x,R_i)}(R_i, R_{N\setminus i})$ for some $x \in A$.

Definition 2.7. A PR Φ is *strategy-proof* if for all $i \in N$, $R_N \in \mathcal{D}^n$, and $R_i' \in \mathcal{D}$, i does not manipulate Φ at R_N via R_i'.

In other words, strategy-proofness of a PR means that a deviation results in a (first order) stochastically dominated lottery for the deviating agent.

For PRs Φ^j, $j = 1, \ldots, k$ and nonnegative numbers λ^j, $j = 1, \ldots, k$, summing to 1, we define the PR $\Phi = \sum_{j=1}^k \Phi^j$ by $\Phi_x(R_N) = \sum_{j=1}^k \lambda^j \Phi_x^j(R_N)$ for all $R_N \in \mathcal{D}^n$ and $x \in A$. We call Φ a *convex combination* of the PRs Φ^j.

Definition 2.8. A domain $\mathcal{D}$ is said to be a *deterministic extreme point* domain if every strategy-proof and unanimous PR on $\mathcal{D}^n$ can be written as a convex combination of strategy-proof and unanimous DRs on $\mathcal{D}^n$.

For $a \in A$, let $\mathcal{D}_a = \{R \in \mathcal{D} : \tau(R) = a\}$.

Definition 2.9. Let $x, y \in A$, $x \neq y$. A domain $\mathcal{D}$ is a *binary restricted domain* over $\{x, y\}$ if

(i) for all $R \in \mathcal{D}$, $\tau(R) \in \{\{x\}, \{y\}, \{x, y\}\}$,
(ii) for all $a, b \in \{x, y\}$ with $a \neq b$, and for each $R \in \mathcal{D}_a$, there exists $R' \in \mathcal{D}_b$ such that $U(b, R) \cap U(a, R') = \{a, b\}$.

Condition (ii) in the definition of a binary restricted domain is used in the proof of Proposition 3.5. There, we also provide an example (see Remark 3.6) to show that this condition cannot be dispensed with.

We conclude this section with the following definition.

Definition 2.10. Let $x, y \in A$, $x \neq y$. A domain $\mathcal{D}$ is a *binary support domain* over $\{x, y\}$ if $\Phi_{\{x,y\}}(R_N) = 1$ for every $R_N \in \mathcal{D}^n$ and every strategy-proof and unanimous PR Φ on $\mathcal{D}^n$.

3. Results

In this section we present the main results of this paper. First we show that every binary support domain is a deterministic extreme point domain (Corollary 3.3). Next we show that every binary restricted domain is a binary support domain (Theorem 3.4). Consequently, every binary restricted domain is a deterministic extreme point domain (Corollary 3.8). Next, we characterize the set of all strategy-proof and unanimous rules on such binary restricted domains.

3.1. Binary support domains are deterministic extreme point domains

First we establish a necessary and sufficient condition for a domain to be a deterministic extreme point domain.

Theorem 3.1. *A domain $\mathcal{D}$ is a deterministic extreme point domain if and only if every strategy-proof and unanimous strict PR on $\mathcal{D}^n$ is a convex combination of two other distinct strategy-proof and unanimous PRs.*

Proof. First, let $\mathcal{D}$ be an arbitrary domain. Observe that every PR Φ can be identified with a vector in $\mathbb{R}^{pm}$, where p is the number of different preference profiles, i.e., the number of elements of $\mathcal{D}^n$, and m is the number of elements of A. Compactness and convexity of a set of PRs are equivalent to convexity and compactness of the associated subset of $\mathbb{R}^{pm}$.

We show that the set of all strategy-proof and unanimous probabilistic rules $\mathcal{S}$ over $\mathcal{D}^n$ is compact and convex.

For convexity, let $\Phi', \Phi'' \in \mathcal{S}$ and $0 \leq \alpha \leq 1$, and let the PR Φ be defined by $\Phi(R_N) = \alpha\Phi'(R_N) + (1-\alpha)\Phi''(R_N)$ for all $R_N \in \mathcal{D}^n$. Clearly, Φ is unanimous. For strategy-proofness, let $i \in N$, $R_N \in \mathcal{D}^n$ and $R_i' \in \mathcal{D}$. Then, for all $b \in A$, by strategy-proofness of Φ' and Φ'' we have $\Phi'_{U(b,R_i)}(R_i', R_{N\setminus i}) \leq \Phi_{U(b,R_i)}(R_N)$ and $\Phi''_{U(b,R_i)}(R_i', R_{N\setminus i}) \leq \Phi''_{U(b,R_i)}(R_N)$, so that

$$\begin{aligned}&\alpha\Phi'_{U(b,R_i)}(R_i', R_{N\setminus i}) + (1-\alpha)\Phi''_{U(b,R_i)}(R_i', R_{N\setminus i})\\ &\quad \leq \alpha\Phi'_{U(b,R_i)}(R_N) + (1-\alpha)\Phi''_{U(b,R_i)}(R_N),\end{aligned}$$

hence $\Phi_{U(b,R_i)}(R_i', R_{N\setminus i}) \leq \Phi_{U(b,R_i)}(R_N)$. Thus, Φ is strategy-proof, and $\mathcal{S}$ is convex.

For closedness, consider a sequence Φ^k, $k \in \mathbb{N}$, in $\mathcal{S}$ such that $\lim_{k\to\infty}\Phi^k = \Phi$, i.e., for all $x \in A$ and $R_N \in \mathcal{D}^n$, $\lim_{k\to\infty}\Phi^k_x(R_N) = \Phi_x(R_N)$. It is easy to see that Φ is unanimous. Suppose that Φ were not strategy-proof. Then there exist $i \in N$, $R_N \in \mathcal{D}^n$ and $R_i' \in \mathcal{D}$ such that for some $b \in A$, $\Phi_{U(b,R_i)}(R_i', R_{N\setminus i}) > \Phi_{U(b,R_i)}(R_N)$. This means there exists $k \in \mathbb{N}$ such that $\Phi^k_{U(b,R_i)}(R_i', R_{N\setminus i}) > \Phi^k_{U(b,R_i)}(R_N)$. This contradicts strategy-proofness of Φ^k. So, $\mathcal{S}$ is closed. Clearly, $\mathcal{S}$ is bounded. Thus, it is compact.

Since $\mathcal{S}$ is compact and convex, by the Theorem of Krein–Milman (e.g., Rockafellar, 1970) it is the convex hull of its (non-empty set of) extreme points. Now, for the if-part of the theorem, for a domain $\mathcal{D}$ satisfying the premise, no strict PR is an extreme point. Thus, $\mathcal{D}$ is a deterministic extreme point domain. In fact, it is also easy to see that every strategy-proof and unanimous deterministic rule is an extreme point of $\mathcal{S}$.

For the only-if part, let $\mathcal{D}$ be a deterministic extreme point domain and let Φ be a strategy-proof and unanimous strict PR on $\mathcal{D}^n$. Then there are $\lambda^1, \ldots, \lambda^k$, $k \geq 2$, with $\lambda^i > 0$ for all $i = 1, \ldots, k$ and $\sum_{i=1}^k \lambda^i = 1$, and strategy-proof and unanimous DRs $f^1, \ldots, f^k$ on $\mathcal{D}^n$ with $f^i \neq f^j$ for $i \neq j$, such that $\Phi = \sum_{i=1}^k \lambda^i f^i$. We define $\Phi' = \sum_{i=2}^k \frac{\lambda^i}{1-\lambda^1} f^i$. Then $\Phi = (1-\lambda^1)\Phi' + \lambda^1 f^1$, and Φ' and f^1 are distinct strategy-proof and unanimous PRs different from Φ. ■

In the following theorem we show that if a strategy-proof and unanimous strict PR has binary support, then it can be written as a convex combination of two other strategy-proof and unanimous PRs.

Theorem 3.2. *Let $\Phi : \mathcal{D}^n \to \Delta(A)$ be a strategy-proof and unanimous strict PR and let $x, y \in A$ such that $\Phi_{\{x,y\}}(R_N) = 1$ for all $R_N \in \mathcal{D}^n$. Then there exist strategy-proof and unanimous PRs Φ', Φ'' with $\Phi' \neq \Phi''$ such that $\Phi(R_N) = \frac{1}{2}\Phi'(R_N) + \frac{1}{2}\Phi''(R_N)$ for all $R_N \in \mathcal{D}^n$.*

Proof. Note that $\Phi_{\{x,y\}}(R_N) = 1$ for all $R_N \in \mathcal{D}^n$ implies that $\Phi(R_N)$ is completely determined by $\Phi_x(R_N)$ for all $R_N \in \mathcal{D}^n$. Since Φ is a strict PR, there exists $R_N' \in \mathcal{D}^n$ such that $\Phi_x(R_N') = p \in (0, 1)$. Let $C = \{R_N \in \mathcal{D}^n : \Phi_x(R_N) \neq p\}$. Since C is finite set, there is an $\epsilon \in (0, p)$ such that for all $R_N \in C$, $\Phi_x(R_N) \notin [p-\epsilon, p+\epsilon]$. We define Φ' and Φ'' with support $\{x, y\}$ by

$$\Phi'_x(R_N) = \begin{cases}\Phi_x(R_N) & \text{if } R_N \in C\\ \Phi_x(R_N)+\epsilon & \text{otherwise}\end{cases} \quad \text{and}$$

$$\Phi''_x(R_N) = \begin{cases}\Phi_x(R_N) & \text{if } R_N \in C\\ \Phi_x(R_N)-\epsilon & \text{otherwise.}\end{cases}$$

Clearly, $\Phi' \neq \Phi''$ and $\Phi(R_N) = \frac{1}{2}\Phi'(R_N) + \frac{1}{2}\Phi''(R_N)$ for all $R_N \in \mathcal{D}^n$. Unanimity of Φ' and Φ'' follows from unanimity of Φ. We show that Φ' and Φ'' are strategy-proof. We consider only Φ', the proof for Φ'' is analogous. Let $i \in N$, $R_N \in \mathcal{D}^n$ and $Q_i \in \mathcal{D}$. Write $Q_N = (Q_i, R_{N\setminus i})$. We consider the following cases.

Case 1 $R_N, Q_N \notin C$. Then $\Phi'_x(R_N) = p+\epsilon = \Phi'_x(Q_N)$. So i does not manipulate Φ' at R_N via Q_i.

Case 2 $R_N, Q_N \in C$. Then $\Phi'_x(R_N) = \Phi_x(R_N)$ and $\Phi'_x(Q_N) = \Phi_x(Q_N)$. Since i does not manipulate Φ at R_N via Q_i, this implies that i does not manipulate Φ' at R_N via Q_i.

Case 3 $R_N \notin C$, $Q_N \in C$. Then $\Phi'_x(R_N) = \Phi_x(R_N) + \epsilon$ and $\Phi'_x(Q_N) = \Phi_x(Q_N) \notin [\Phi_x(R_N)-\epsilon, \Phi_x(R_N)+\epsilon]$. If xP_iy (where P_i is the asymmetric part of R_i), then by strategy-proofness of Φ, $\Phi'_x(Q_N) = \Phi_x(Q_N) \leq \Phi_x(R_N) = \Phi'_x(R_N) - \epsilon < \Phi'_x(R_N)$, so that i does not manipulate Φ' at R_N via Q_i. If yP_ix, then by strategy-proofness of Φ, $\Phi'_x(Q_N) = \Phi_x(Q_N) \geq \Phi_x(R_N) + \epsilon = \Phi'_x(R_N)$, so that i does not manipulate Φ' at R_N via Q_i.

Case 4 $R_N \in C$, $Q_N \notin C$. If xP_iy then by strategy-proofness of Φ and the choice of ϵ, $\Phi'_x(Q_N) = \Phi_x(Q_N) + \epsilon \leq (\Phi_x(R_N) - \epsilon) + \epsilon = \Phi_x(R_N) = \Phi'_x(R_N)$, so that i does not manipulate Φ' at R_N via Q_i. If yP_ix, then by strategy-proofness of Φ, $\Phi'_y(Q_N) = \Phi_y(Q_N) - \epsilon \leq \Phi_y(R_N) - \epsilon = \Phi'_y(R_N) - \epsilon < \Phi'_y(R_N)$, so that i does not manipulate Φ' at R_N via Q_i. ■

Theorems 3.2 and 3.1 imply the following result.

Corollary 3.3. *Every binary support domain is a deterministic extreme point domain.*

3.2. Binary restricted domains are binary support domains

The main result of this subsection is the following theorem.

Theorem 3.4. *Every binary restricted domain is a binary support domain.*

We first prove the result for two agents and then use induction to prove it for an arbitrary number of agents.

Proposition 3.5. *Let $\mathcal{D}$ be a binary restricted domain over $\{x, y\}$, and let $\Phi : \mathcal{D}^2 \to \triangle A$ be a strategy-proof and unanimous PR. Then $\Phi_{\{x,y\}}(R_N) = 1$ for all $R_N \in \mathcal{D}^2$.*

Proof. By unanimity of Φ it is sufficient to consider the case where $R_N = (R_1, R_2)$ with $R_1 \in \mathcal{D}_x$ and $R_2 \in \mathcal{D}_y$.

First assume that $U(y, R_1) \cap U(x, R_2) = \{x, y\}$. Suppose that $\Phi_B(R_N) > 0$ for $B = A \setminus U(y, R_1)$. Then agent 1 manipulates at R_N via some $R_1' \in \mathcal{D}_y$, since by unanimity $\Phi_y(R_1', R_2) = 1$ and y is strictly preferred to (every element of) $A \setminus U(y, R_1)$ at the preference R_1 of agent 1. Hence, we must have $\Phi_B(R_N) = 0$ for $B = A \setminus U(y, R_1)$. Similarly one shows that $\Phi_{B'}(R_N) = 0$ for $B' = A \setminus U(x, R_2)$. Since $U(y, R_1) \cap U(x, R_2) = \{x, y\}$, we have $\Phi_{\{x,y\}}(R_N) = 1$.

Next, suppose that $U(y, R_1) \cap U(x, R_2) \neq \{x, y\}$. This, by the definition of a binary restricted domain, means that there exist $R_1' \in \mathcal{D}_x$ and $R_2' \in \mathcal{D}_y$ such that $U(y, R_1) \cap U(x, R_2') = \{x, y\}$ and

$U(y, R_1') \cap U(x, R_2) = \{x, y\}$. By the first part of the proof we have $\Phi_{\{x,y\}}(R_1, R_2') = 1$ and $\Phi_{\{x,y\}}(R_1', R_2) = 1$. Let $\Phi_x(R_1, R_2') = \epsilon$ and $\Phi_x(R_1', R_2) = \epsilon'$. Since $R_1, R_1' \in \mathcal{D}_x$ and $R_2, R_2' \in \mathcal{D}_y$, strategy-proofness implies $\Phi_x(R_1', R_2') = \Phi_x(R_1, R_2') = \epsilon$ and $\Phi_y(R_1', R_2') = \Phi_y(R_1', R_2) = 1 - \epsilon'$. This means $\Phi_{\{x,y\}}(R_1', R_2') = \epsilon + 1 - \epsilon'$, which implies $\epsilon \leq \epsilon'$. By a similar argument it follows that $\epsilon' \leq \epsilon$. Hence, $\epsilon = \epsilon'$. Finally, again since $R_1, R_1' \in \mathcal{D}_x$ and $R_2, R_2' \in \mathcal{D}_y$, we have by strategy-proofness that $\Phi_x(R_1, R_2) = \Phi_x(R_1', R_2) = \epsilon$ and $\Phi_y(R_1, R_2) = \Phi_y(R_1, R_2') = 1 - \epsilon$, and hence $\Phi_{\{x,y\}}(R_1, R_2) = 1$, completing the proof. ■

Remark 3.6. Condition (ii) in Definition 2.9 of a binary restricted domain cannot be omitted. Let $A = \{x, y, z\}$, $N = \{1, 2\}$, and let $\mathcal{D} = \{R, R'\} \subseteq \mathbb{W}(A)$ with $xPzPy$ and $yP'zP'x$ (P and P' are the asymmetric parts of R and R', respectively). Hence, $\mathcal{D}$ is not a binary restricted domain over $\{x, y\}$, since (ii) in Definition 2.9 is not fulfilled. Let $(\alpha, \beta, \gamma) \in \Delta(A)$ be the lottery with probabilities on x, y, and z, respectively. Define the PR Φ by: $\Phi(R_N) = (1, 0, 0)$ if $R_N = (R, R)$, $\Phi(R_N) = (0, 1, 0)$ if $R_N = (R', R')$, and $\Phi(R_N) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ in the two other cases. Then clearly Φ is unanimous and strategy-proof. Hence, $\mathcal{D}$ is not a binary support domain.

The following proposition treats the case with more than two agents.

Proposition 3.7. *Let $n \geq 3$, let $\mathcal{D}$ be binary restricted domain over $\{x, y\}$, and let $\Phi : \mathcal{D}^n \to \triangle A$ be a strategy-proof and unanimous PR. Then $\Phi_{\{x,y\}}(R_N) = 1$ for all $R_N \in \mathcal{D}^n$.*

Proof. As before, $N = \{1, \ldots, n\}$ is the set of agents. We prove the result by induction. Assume that the proposition holds for all sets with $k < n$ agents.

Let $N^* = \{1, 3, \ldots, n\}$ and define the PR $g : \mathcal{D}^{n-1} \to \triangle A$ for the set of agents N^* as follows: For all $R_{N^*} = (R_1, R_3, \ldots, R_n) \in \mathcal{D}^{n-1}$,

$$g(R_1, R_3, \ldots, R_n) = \Phi(R_1, R_1, R_3, \ldots, R_n).$$

Claim 1. $g_{\{x,y\}}(R_{N^*}) = 1$ *for all* $R_{N^*} \in \mathcal{D}^{n-1}$.

To prove this claim, first observe that g inherits unanimity from Φ. We show that g also inherits strategy-proofness. It is easy to see that agents other than 1 do not manipulate g since Φ is strategy-proof. Let $(R_1, R_3, \ldots, R_n) \in \mathcal{D}^{n-1}$ and $Q_1 \in \mathcal{D}$. For all $b \in A$, we have

$$\begin{aligned} g_{U(b,R_1)}(R_1, R_3, \ldots, R_n) &= \Phi_{U(b,R_1)}(R_1, R_1, R_3, \ldots, R_n) \\ &\geq \Phi_{U(b,R_1)}(Q_1, R_1, R_3, \ldots, R_n) \\ &\geq \Phi_{U(b,R_1)}(Q_1, Q_1, R_3, \ldots, R_n) \\ &= g_{U(b,R_1)}(Q_1, R_3, \ldots, R_n), \end{aligned}$$

where the inequalities follow from strategy-proofness of Φ. The proof of Claim 1 is now complete by the induction hypothesis.[3]

Thus, by Claim 1, we have $\Phi_{\{x,y\}}(R_N) = 1$ for all $R_N \in \mathcal{D}^n$ with $R_1 = R_2$. Our next claim shows that the same holds if $\tau(R_1) = \tau(R_2)$.

Claim 2. *Let R_N be a preference profile such that $\tau(R_1) = \tau(R_2)$. Then $\Phi_{\{x,y\}}(R_N) = 1$.*

To prove this claim, first suppose that $\tau(R_1) = \tau(R_2) = \{x, y\}$. Then, if $\Phi_{\{x,y\}}(R_N) < 1$, player 1 manipulates at R_N via R_2 since by Claim 1, $\Phi_{\{x,y\}}(R_2, R_2, R_{N\setminus\{1,2\}}) = 1$. Now consider the case $\tau(R_1) = \tau(R_2) \in \{x, y\}$, say $\tau(R_1) = \tau(R_2) = x$. By Claim 1 we have $\Phi_{\{x,y\}}(R_1, R_1, R_{N\setminus\{1,2\}}) = \Phi_{\{x,y\}}(R_2, R_2, R_{N\setminus\{1,2\}}) = 1$. Moreover, since $\tau(R_1) = \tau(R_2) = x$ we have by strategy-proofness $\Phi_x(R_1, R_1, R_{N\setminus\{1,2\}}) = \Phi_x(R_1, R_2, R_{N\setminus\{1,2\}}) = \Phi_x(R_2, R_2, R_{N\setminus\{1,2\}}) = \epsilon$ (say).

Since $\mathcal{D}$ is a binary restricted domain, if $\tau(R_i) \neq y$ for all $i \in N \setminus \{1, 2\}$, then by unanimity $\Phi_{\{x,y\}}(R_N) = \Phi_x(R_N) = 1$, and we are done. Now suppose there is $i \in N \setminus \{1, 2\}$ such that $\tau(R_i) = y$. Let $R \in \mathcal{D}$ be such that $\tau(R) = y$ and $U(x, R) \cap U(y, R_1) = \{x, y\}$. Such an R exists since $\mathcal{D}$ is a binary restricted domain. Consider the preference profile $\bar{R}_{N\setminus\{1,2\}}$ of the agents in $N \setminus \{1, 2\}$ defined as follows: for all $i \in N \setminus \{1, 2\}$

$$\bar{R}_i = \begin{cases} R & \text{if } \tau(R_i) = y \\ R_i & \text{otherwise.} \end{cases}$$

By Claim 1, $\Phi_{\{x,y\}}(R_1, R_1, \bar{R}_{N\setminus\{1,2\}}) = \Phi_{\{x,y\}}(R_2, R_2, \bar{R}_{N\setminus\{1,2\}}) = 1$. Since $\tau(R_1) = \tau(R_2) = x$, we have by strategy-proofness $\Phi_x(R_1, R_1, \bar{R}_{N\setminus\{1,2\}}) = \Phi_x(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) = \Phi_x(R_2, R_2, \bar{R}_{N\setminus\{1,2\}})$. We show $\Phi_x(R_1, R_1, \bar{R}_{N\setminus\{1,2\}}) = \epsilon$. First we claim that $\Phi_y(R_1, R_1, R_{N\setminus\{1,2\}}) = \Phi_y(R_1, R_1, \bar{R}_{N\setminus\{1,2\}})$. To see this, consider a player $i \in N \setminus \{1, 2\}$ such that $R_i \neq \bar{R}_i$. Then $\tau(R_i) = \tau(\bar{R}_i) = y$, hence by strategy-proofness we have $\Phi_y(R_1, R_1, R_i, R_{N\setminus\{1,2,i\}}) = \Phi_y(R_1, R_1, \bar{R}_i, R_{N\setminus\{1,2,i\}})$. By repeating this argument, $\Phi_y(R_1, R_1, R_{N\setminus\{1,2\}}) = \Phi_y(R_1, R_1, \bar{R}_{N\setminus\{1,2\}})$. Hence, since $\Phi_{\{x,y\}}(R_1, R_1, \bar{R}_{N\setminus\{1,2\}}) = 1$, we obtain $\Phi_x(R_1, R_1, \bar{R}_{N\setminus\{1,2\}}) = \epsilon$.

Using similar logic it follows that $\Phi_y(R_1, R_2, R_{N\setminus\{1,2\}}) = \Phi_y(R_1, R_2, \bar{R}_{N\setminus\{1,2\}})$. We complete the proof by showing $\Phi_y(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) = 1-\epsilon$. For this, since $\Phi_x(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) = \epsilon$, it suffices to show that $\Phi_{\{x,y\}}(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) = 1$. Suppose that $\Phi_B(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) > 0$ for $B = A \setminus U(y, R_1)$. Then agent 1 manipulates at $(R_1, R_2, \bar{R}_{N\setminus\{1,2\}})$ via R_2 since $\Phi_{\{x,y\}}(R_2, R_2, \bar{R}_{N\setminus\{1,2\}}) = 1$. Thus, $\Phi_{U(y,R_1)}(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) = 1$. Next we show that $\Phi_{U(x,R)}(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) = 1$. If not, consider $i \in N \setminus \{1, 2\}$ such that $\bar{R}_i = R$. Let R_i' be such that $\tau(R_i') = x$. Then by strategy-proofness $\Phi_{U(x,R)}(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) \geq \Phi_{U(x,R)}(R_1, R_2, R_i', \bar{R}_{N\setminus\{1,2,i\}})$. By sequentially changing the preferences of the players in $N \setminus \{1, 2\}$ with y at the top in this manner we construct a preference profile $\hat{R}$ such that $\tau(\hat{R}_i) = x$ for all $i \in N$ and $\Phi_{U(x,R)}(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) \geq \Phi_{U(x,R)}(\hat{R}) = 1$. Hence $\Phi_{U(x,R)}(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) = 1$.

Since $\Phi_{U(y,R_1)}(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) = 1$, $\Phi_{U(x,R)}(R_1, R_2, \bar{R}_{N\setminus\{1,2\}}) = 1$, and $U(y, R_1) \cap U(x, R) = \{x, y\}$, we have $\Phi_{\{x,y\}}(R_1, R_2, R_{N\setminus\{1,2\}}) = 1$. This completes the proof of Claim 2.

We can now complete the proof of the proposition. Let $R_N \in \mathcal{D}^n$ be an arbitrary preference profile. We show that $\Phi_{\{x,y\}}(R_N) = 1$. In view of Claim 2, we may assume $\tau(R_1) \neq \tau(R_2)$. Note that if $\tau(R_i) = \{x, y\}$ for some $i \in \{1, 2\}$ and $\Phi_{A\setminus\{x,y\}}(R_N) > 0$, then agent i manipulates at R_N via R_j, where $j \in \{1, 2\}, j \neq i$, since by Claim 1 we have $\Phi_{\{x,y\}}(R_j, R_j, R_{N\setminus\{1,2\}}) = 1$. So we may assume without loss of generality that $\tau(R_1) = x$ and $\tau(R_2) = y$.

Suppose $U(y, R_1) \cap U(x, R_2) = \{x, y\}$. If $\Phi_{A\setminus U(x,R_2)}(R_N) > 0$, then agent 2 manipulates at R_N via R_1 since, by Claim 1, $\Phi_{\{x,y\}}(R_1, R_1, R_{N\setminus\{1,2\}}) = 1$. Thus, $\Phi_{U(x,R_2)}(R_N) = 1$, and similarly one proves $\Phi_{U(y,R_1)}(R_N) = 1$. Together with $U(y, R_1) \cap U(x, R_2) = \{x, y\}$, this implies $\Phi_{\{x,y\}}(R_N) = 1$.

Finally, suppose $U(y, R_1) \cap U(x, R_2) \neq \{x, y\}$. Since $\mathcal{D}$ is a binary restricted domain there exist $R_1' \in \mathcal{D}_x$ and $R_2' \in \mathcal{D}_y$ such that $U(y, R_1) \cap U(x, R_2') = \{x, y\}$ and $U(y, R_1') \cap U(x, R_2) = \{x, y\}$. Since $\tau(R_1) = \tau(R_1') = x$ and $\tau(R_2) = \tau(R_2') = y$, by strategy-proofness we have $\Phi_x(R_1, R_2, R_{N\setminus\{1,2\}}) = \Phi_x(R_1', R_2, R_{N\setminus\{1,2\}})$ and $\Phi_y(R_1, R_2, R_{N\setminus\{1,2\}}) = \Phi_y(R_1, R_2', R_{N\setminus\{1,2\}})$. By a similar argument as in the last paragraph of proof of Proposition 3.5 we have $\Phi_x(R_1, R_2', R_{N\setminus\{1,2\}}) = \Phi_x(R_1', R_2, R_{N\setminus\{1,2\}})$.

[3] We have included the proof of Claim 1 for completeness. It can also be found in Sen (2011).

Hence, $\Phi_{\{x,y\}}(R_1, R_2, R_{N\setminus\{1,2\}}) = \Phi_{\{x,y\}}(R_1, R'_2, R_{N\setminus\{1,2\}})$. However, $\Phi_{\{x,y\}}(R_1, R'_2, R_{N\setminus\{1,2\}}) = 1$ since $U(y, R_1) \cap U(x, R'_2) = \{x, y\}$, which completes the proof of the proposition. ■

Theorem 3.4 now follows from Propositions 3.5 and 3.7. Moreover, we have the following consequence of Theorem 3.4 and Corollary 3.3.

Corollary 3.8. *Every binary restricted domain is a deterministic extreme point domain.*

3.3. Characterization of strategy-proof and unanimous rules

In this subsection we give a characterization of all strategy-proof and unanimous PRs on a binary restricted domain. In view of Corollary 3.8, it will be sufficient to give a characterization of strategy-proof and unanimous DRs on a binary restricted domain.

Throughout this subsection let $\mathcal{D}$ be a binary restricted domain over $\{x, y\}$. For $R_N \in \mathcal{D}^n$, by $N^x(R_N)$ we denote the set of agents $i \in N$ such that $\tau(R_i) = x$; by $N^{xy}(R_N)$ the set of agents $i \in N$ such that $\tau(R_i) = \{x, y\}$; and we define

$$\mathcal{I}(R_N) = \{Q_N \in \mathcal{D}^n : N^{xy}(Q_N) = N^{xy}(R_N) \text{ and } R_i = Q_i \text{ for every } i \in N^{xy}(R_N)\}.$$

Thus, $\mathcal{I}(R_N)$ is the (equivalence) class of all preference profiles that share with R_N the set of agents who are indifferent between x and y and have the same preference as in R_N.

For $R_N \in \mathcal{D}^N$ a *committee* $\mathcal{W}(R_N)$ is a set of subsets of N such that:

(1) If $N^{xy}(R_N) = N$ then $\mathcal{W}(R_N) = \emptyset$ or $\mathcal{W}(R_N) = \{\emptyset\}$.
(2) If $N^{xy}(R_N) \neq N$ then $\mathcal{W}(R_N) \subseteq 2^{N\setminus N^{xy}(R_N)}$ satisfies
  (i) $\emptyset \notin \mathcal{W}(R_N)$ and $N \setminus N^{xy}(R_N) \in \mathcal{W}(R_N)$,
  (ii) for all $S, T \subseteq N \setminus N^{xy}(R_N)$, if $S \subseteq T$ and $S \in \mathcal{W}(R_N)$, then $T \in \mathcal{W}(R_N)$.

In case (2) in the above definition, a committee is a simple game, elements of $\mathcal{W}(R_N)$ are called *winning coalitions*, and other subsets of $N \setminus N^{x,y}(R_N)$ are called *losing coalitions*.

A collection of committees $\mathcal{W} = \{\mathcal{W}(R_N) : R_N \in \mathcal{D}^n\}$ is an *admissible collection of committees* (ACC) if the following three conditions hold:

(a) For all $R_N, Q_N \in \mathcal{D}^n$, if $Q_N \in \mathcal{I}(R_N)$ then $\mathcal{W}(Q_N) = \mathcal{W}(R_N)$.
(b) For all $R_N \in \mathcal{D}^n$, $i \in N \setminus N^{xy}(R_N)$, $R'_i \in \mathcal{D}$ such that $\tau(R'_i) = \{x, y\}$, and $C \in \mathcal{W}(R_N)$, if $i \notin C$, then $C \in \mathcal{W}(R_{N\setminus i}, R'_i)$.
(c) For all $R_N \in \mathcal{D}^n$, $i \in N \setminus N^{xy}(R_N)$, $R'_i \in \mathcal{D}$ such that $\tau(R'_i) = \{x, y\}$, and $C \notin \mathcal{W}(R_N)$, if $i \in C$, then $C \setminus \{i\} \notin \mathcal{W}(R_{N\setminus i}, R'_i)$.

Thus, a collection of committees is admissible if (a) each committee depends only on the set of indifferent agents and their preferences; (b) if a coalition is winning and an agent not belonging to it becomes indifferent, then the coalition stays winning; and (c) if a coalition is losing and an agent belonging to it becomes indifferent, then the coalition without that agent stays losing. Observe that (a), (b), and (c) are trivially fulfilled if $N^{xy}(R_N) = N$, i.e., if all agents are indifferent. In particular, in that case $\mathcal{I}(R_N) = \{R_N\}$.

With an ACC $\mathcal{W}$ we associate a DR $f_{\mathcal{W}}$ as follows: for every $R_N \in \mathcal{D}^n$,

$$f_{\mathcal{W}}(R_N) = \begin{cases} x & \text{if } N^x(R_N) \in \mathcal{W}(R_N) \\ y & \text{if } N^x(R_N) \notin \mathcal{W}(R_N). \end{cases}$$

We now show that every strategy-proof and unanimous DR is of the form $f_{\mathcal{W}}$. We just outline the proof since it is rather standard, and, moreover, the theorem is almost equivalent to Theorem 3 in Larsson and Svensson (2006). A (nonessential) difference is that the last mentioned result is formulated for the case where $A = \{x, y\}$, so that all preference profiles with the same indifferent agents are equivalent, making our condition (a) on an ACC redundant.

Theorem 3.9. *Let $\mathcal{D}$ be a binary restricted domain. A DR f on $\mathcal{D}^n$ is strategy-proof and unanimous if and only if there is an ACC $\mathcal{W}$ such that $f = f_{\mathcal{W}}$.*

Proof. For the only-if part, let f be a strategy-proof and unanimous DR. For each $R_N \in \mathcal{D}^n$ we define the set $\mathcal{W}_f(R_N)$ of coalitions as follows. If $N^{xy}(R_N) = N$ then $\mathcal{W}_f(R_N) = \{\emptyset\}$ if $f(R_N) = x$ and $\mathcal{W}_f(R_N) = \emptyset$ otherwise. If $N^{xy}(R_N) \neq N$ then for every $C \subseteq N \setminus N^{xy}(R_N)$, $C \in \mathcal{W}_f(R_N)$ if and only if there is a $Q_N \in \mathcal{I}(R_N)$ such that $f(Q_N) = x$ and $C = N^x(Q_N)$. Then $\mathcal{W}_f(R_N)$ is a committee for each $R_N \in \mathcal{D}^N$ by unanimity and strategy-proofness of f. Also, the collection $\mathcal{W}_f = \{\mathcal{W}_f(R_N) : R_N \in \mathcal{D}^n\}$ is an ACC: (a) follows directly by definition of the committees $\mathcal{W}_f(R_N)$; and (b) and (c) follow from unanimity and strategy-proofness of f. Finally, it is straightforward to check that $f = f_{\mathcal{W}_f}$.

For the if-part, let $\mathcal{W}$ be an ACC. Then it is easy to check that $f = f_{\mathcal{W}}$ is strategy-proof and unanimous. ■

By Corollary 3.8 and Theorem 3.9 we obtain the following result.

Corollary 3.10. *Let $\mathcal{D}$ be a binary restricted domain. A PR f on $\mathcal{D}^n$ is strategy-proof and unanimous if and only if it is a convex combination of DRs of the form $f = f_{\mathcal{W}}$ for ACCs $\mathcal{W}$.*

Remark 3.11. The set of winning coalitions $\mathcal{W}(R_N)$ may indeed depend on the preference profile of the indifferent agents, i.e., the agents in $\mathcal{I}(R_N)$. Here is an example. Let $N = \{1, 2, 3\}$, $A = \{x, y, v, w\}$ and define: $\mathcal{W}(R_N) = \{\{1, 3\}, N\}$ if $N^{xy}(R_N) = \emptyset$; $\mathcal{W}(R_N) = \{\{1, 3\}\}$ if $N^{xy}(R_N) = \{2\}$; $\mathcal{W}(R_N) = \{\{2, 3\}\}$ if $N^{xy}(R_N) = \{1\}$; $\mathcal{W}(R_N) = \{\{1, 2\}\}$ if $N^{xy}(R_N) = \{3\}$ and vR_3w; $\mathcal{W}(R_N) = \{\{1\}, \{1, 2\}\}$ if $N^{xy}(R_N) = \{3\}$ and wP_3v; and $\mathcal{W}(R_N) = \{\emptyset\}$ if $N^{xy}(R_N) = N$. Then it is straightforward to verify that $f_{\mathcal{W}}$ is strategy-proof and unanimous.

4. Application to single-dipped preferences

In this section we apply our results to single-dipped domains and characterize all strategy-proof and unanimous PRs on such a domain.

Definition 4.1. A preference of agent $i \in N$, $R_i \in \mathbb{W}(A)$, is *single-dipped* on A relative to a linear ordering $\succ$ of the set of alternatives if

(i) R_i has a unique minimal element $d(R_i)$, the *dip* of R_i and
(ii) for all $y, z \in A$, $[d(R_i) \succeq y \succ z$ or $z \succ y \succeq d(R_i)] \Rightarrow zP_iy$.

Let $\mathcal{D}_\succ$ denote the set of all single-dipped preferences relative to the ordering $\succ$, and let $\mathcal{R}_\succ \subseteq \mathcal{D}_\succ$. Clearly $\mathcal{D}_\succ$ is a binary restricted domain. Moreover, $\mathcal{R}_\succ$ is a binary restricted domain if it satisfies condition (ii) in Definition 2.9, the definition of a binary restricted domain. Hence, by Corollary 3.8 and Theorem 3.9 we obtain the following result.

Corollary 4.2. *Let $\succ$ be a linear ordering over A and let $\mathcal{R}_\succ \subseteq \mathcal{D}_\succ$ satisfy* (ii) *in Definition 2.9. Then a PR on $\mathcal{R}^n_\succ$ is strategy-proof and unanimous if and only if it is a convex combination of DRs on $\mathcal{R}^n_\succ$ of the form $f = f_{\mathcal{W}}$ for ACCs $\mathcal{W}$.*

Consider a single-dipped domain where the alternatives are assumed to be equidistant from each other and preference is consistent with the distance from the dip. More precisely, when the distance of an alternative from the dip of an agent is higher than that of another alternative, the agent prefers the former alternative to the latter. Call such a domain a 'distance single-dipped domain'. If ties between equidistant alternatives are broken in both ways, then such a domain is again a binary restricted domain, and Corollary 4.2 applies. However, if ties are broken in favor of the left side (or of the right side) only, then the domain is no longer a binary restricted domain. Indeed, in Example 4.3 we show that there exists a strategy-proof and unanimous PR that does not have binary support.

Example 4.3. Consider the distance single-dipped domain presented in the table below. There are two agents and four alternatives: think of the alternatives as located on a line in the ordering $x_1 < x_2 < x_3 < x_4$ with equal distances. Ties are always broken in favor of the left alternative. It is not hard to verify that the PR given in the table (probabilities in the order x_1, x_2, x_3, x_4, and $0 < \beta < \alpha < 1, 0 < \gamma < \epsilon < 1$ arbitrary) is strategy-proof and unanimous, but does not have binary support.

1/2	$x_1x_2x_3x_4$	$x_4x_3x_2x_1$	$x_4x_1x_3x_2$	$x_1x_2x_4x_3$
$x_1x_2x_3x_4$	$(1, 0, 0, 0)$	$(\alpha - \beta, \beta, 0, 1 - \alpha)$	$(\alpha, 0, 0, 1 - \alpha)$	$(1, 0, 0, 0)$
$x_4x_3x_2x_1$	$(\epsilon - \gamma, \gamma, 0, 1 - \epsilon)$	$(0, 0, 0, 1)$	$(0, 0, 0, 1)$	$(\epsilon - \gamma, \gamma, 0, 1 - \epsilon)$
$x_4x_1x_3x_2$	$(\epsilon, 0, 0, 1 - \epsilon)$	$(0, 0, 0, 1)$	$(0, 0, 0, 1)$	$(\epsilon, 0, 0, 1 - \epsilon)$
$x_1x_2x_4x_3$	$(1, 0, 0, 0)$	$(\alpha - \beta, \beta, 0, 1 - \alpha)$	$(\alpha, 0, 0, 1 - \alpha)$	$(1, 0, 0, 0)$

Remark 4.4. Other examples of binary restricted domains are single-peaked domains where each peak can only be one of two fixed adjacent alternatives, or certain single-crossing domains with only two alternatives that can serve as top alternative. These domains, however, are of limited interest within the single-peaked and single-crossing domains, respectively.

Of course, there are binary restricted domains which are much larger than and considerably different from single-dipped domains—an obvious example is the domain of all preferences with x or y or both on top, or any subdomain including a preference with x on top and y second and a preference with y on top and x second.

5. Infinitely many alternatives

In this section we assume that the set of alternatives A may be an infinite set, for instance a closed interval in $\mathbb{R}$. We assume A to be endowed with a σ-algebra of measurable sets; only preferences in $\mathbb{W}(A)$ for which the upper contour sets $U(x, R)$, $x \in A$, are measurable, are considered. A PR Φ assigns to an admissible preference profile a probability distribution over the measurable space A, hence a probability to every measurable set. The set of all such probability distributions will still be denoted as $\Delta(A)$. For a measurable set $B \subseteq A$, $\Phi_B(R_N)$ denotes the probability assigned to B if the preference profile is R_N. All the introduced concepts and definitions extend in a straightforward manner to this setting. In particular, Definitions 2.1–2.7, 2.9 and 2.10 are literally the same. Also Propositions 3.5 and 3.7 are still valid, and therefore Theorem 3.4 still holds: a binary restricted domain over $\{x, y\}$ ($x, y \in A$) is a binary support domain. The purpose of this section is to provide a characterization of all strategy-proof and unanimous PRs on a binary restricted domain.

Let $\mathcal{D}$ be a binary restricted domain over $\{x, y\}$ for some $x, y \in A$. We use some of the notations introduced in Section 3.3. For $R_N \in \mathcal{D}^n$ with $N^{xy}(R_N) = N$ we let $h(R_N) = h(R_N)(\emptyset) \in [0, 1]$ and for $R_N \in \mathcal{D}^n$ with $N^{xy}(R_N) \neq N$ we let $h(R_N) : 2^{N \setminus N^{xy}(R_N)} \to [0, 1]$ satisfy $h(R_N)(\emptyset) = 0$, $h(R_N)(N \setminus N^{xy}(R_N)) = 1$, and $h(R_N)(C) \leq h(R_N)(C')$ for all $C, C' \subseteq N \setminus N^{xy}(R_N)$ with $C \subseteq C'$; we assume, moreover, that $h(Q_N) = h(R_N)$ whenever $Q_N \in \mathcal{I}(R_N)$ and that

$$h(R_N)(C \setminus i) \leq h(R'_N)(C \setminus i) \leq h(R_N)(C)$$

whenever $i \in N \setminus N^{xy}(R_N)$, $R'_N = (R_{N\setminus i}, R'_i)$ for some R'_i with $\tau(R'_i) = \{x, y\}$, and $C \subseteq N \setminus N^{xy}(R_N)$ with $i \in C$. Observe that such an h generalizes the concept of an admissible collection of committees: we call h a *probabilistic admissible collection of committees* (PACC). For $R_N \in \mathcal{D}^n$ with $N^{xy}(R_N) \neq N$, the number $h(R_N)(C)$ can be interpreted as the probability that a coalition C is winning given a profile with $N^{xy}(R_N)$ as the set of agents who are indifferent between x and y and having $R_{N^{xy}(R_N)}$ as preference profile; specifically, if C is the set of agents with x on top, then this probability will be assigned to x. If $N^{xy}(R_N) = N$, then $h(R_N) = h(R_N)(\emptyset)$ is the probability assigned to x.

We say that a PR Φ on $\mathcal{D}^n$ *is associated with a* PACC h if (i) $\Phi_{\{x,y\}}(R_N) = 1$ for all $R_N \in \mathcal{D}^n$; (ii) $\Phi_x(R_N) = h(R_N)(N^x(R_N))$ for all $R_N \in \mathcal{D}^n$.

We have the following result.

Theorem 5.1. *Let $\mathcal{D}$ be a binary restricted domain over $\{x, y\}$. A PR Φ on $\mathcal{D}^n$ is strategy-proof and unanimous if and only if it is associated with a PACC.*

Proof. For the if-part, let PR Φ be associated with a PACC h. We show that Φ is unanimous and strategy-proof.

We first show that Φ is unanimous. Consider a profile $R_N \in \mathcal{D}^n$ such that $\cap_{i\in N} \tau(R_i) \neq \emptyset$. If $\tau(R_i) = \{x, y\}$ for all $i \in N$ then unanimity holds by definition. Suppose $\cap_{i\in N} \tau(R_i) = x$. Then $N^x(R_N) = N \setminus N^{xy}(R_N)$. Since $h(R_N)(N \setminus N^{xy}(R_N)) = 1$, we have $\Phi_x(R_N) = 1$. If $\cap_{i\in N} \tau(R_i) = y$ then $N^x(R_N) = \emptyset$ which implies $\Phi_x(R_N) = h(R_N)(\emptyset) = 0$. So, $\Phi_y(R_N) = 1$.

Next we show that Φ is strategy-proof. Consider a profile $R_N \in \mathcal{D}^n$. We only need to consider $i \in N \setminus N^{xy}(R_N)$. Let $R'_i \in \mathcal{D}$ and write $R'_N = (R_{N\setminus i}, R'_i)$. We distinguish four cases and each time show that i cannot improve by R'_i. (i) If $\tau(R_i) = x$ and $\tau(R'_i) = y$ then $\Phi_x(R_N) = h(R_N)(N^x(R_N)) \geq h(R'_N)(N^x(R_N)\setminus i) = h(R'_N)(N^x(R'_N)) = \Phi_x(R'_N)$ by definition of h. (ii) If $\tau(R_i) = y$ and $\tau(R'_i) = x$ then $\Phi_x(R_N) = h(R_N)(N^x(R_N)) \leq h(R'_N)(N^x(R_N)) = h(R'_N)(N^x(R'_N)) = \Phi_x(R'_N)$. This implies $\Phi_y(R_N) \geq \Phi_y(R'_N)$. (iii) If $\tau(R_i) = x$ and $\tau(R'_i) = \{x, y\}$, then, since $N^x(R_N) \setminus i = N^x(R'_i, R_{N\setminus i})$, we have $\Phi_x(R_N) = h(R_N)(N^x(R_N)) \geq h(R'_N)(N^x(R'_N)) = \Phi_x(R'_N)$. (iv) Finally, if $\tau(R_i) = y$ and $\tau(R'_i) = \{x, y\}$, then $\Phi_x(R_N) = h(R_N)(N^x(R_N)) \leq h(R'_N)(N^x(R'_N)) = \Phi_x(R'_N)$, which implies $\Phi_y(R'_N) \leq \Phi_y(R_N)$. This completes the proof that Φ is strategy-proof.

For the only-if part, consider a unanimous and strategy-proof PR Φ on $\mathcal{D}^n$. Then $\Phi_{\{x,y\}}(R_N) = 1$ for all $R_N \in \mathcal{D}^n$ by (the modified version of) Theorem 3.4. We show that Φ is associated with a PACC h. If $R_N \in \mathcal{D}^n$ with $N^{xy}(R_N) = N$, then we define $h(R_N) = h(R_N)(\emptyset) = \Phi_x(R_N)$. Now let $R_N \in \mathcal{D}^n$ with $N^{xy}(R_N) \neq N$. By strategy-proofness, $\Phi(Q_N) = \Phi(R_N)$ for all $Q_N \in \mathcal{D}^n$ with $Q_N \in \mathcal{I}(R_N)$ and $N^x(Q_N) = N^x(R_N)$. Therefore, we can define $h(R_N)(C) = \Phi_x(Q_N)$ for any $Q_N \in \mathcal{I}(R_N)$ such that $C = N^x(Q_N)$. By unanimity of Φ, $h(R_N)(\emptyset) = 0$ and $h(R_N)(N \setminus N^{xy}(R_N)) = 1$. By strategy-proofness, $h(R_N)(C) \leq h(R_N)(C')$ for all $C, C' \subseteq N \setminus N^{xy}(R_N)$ with $C \subseteq C'$.

Clearly, $h(Q_N) = h(R_N)$ whenever $R_N \in \mathcal{D}^n$ and $Q_N \in \mathcal{I}(R_N)$.

Let $R_N \in \mathcal{D}^n$, $i \in N \setminus N^{xy}(R_N)$, $R'_N = (R_{N\setminus i}, R'_i)$ for some R'_i with $\tau(R'_i) = \{x, y\}$, and $C \subseteq N \setminus N^{xy}(R_N)$ with $i \in C$. Consider $Q_N \in \mathcal{I}(R_N)$ with $N^x(Q_N) = C$. Then by strategy-proofness we have $h(R_N)(C) = \Phi_x(Q_N) \geq \Phi_x(Q_{N\setminus i}, R'_i) = h(R_{N\setminus i}, R'_i)(C \setminus i) = h(R'_N)(C \setminus i)$. Finally, consider $V_N \in \mathcal{I}(R_N)$ with $N^x(V_N) = C \setminus i$. Again by strategy-proofness we obtain $h(R_N)(C \setminus i) = \Phi_x(V_N) \leq \Phi_x(V_{N\setminus i}, R'_i) = h(R_{N\setminus i}, R'_i)(C \setminus i) = h(R'_N)(C \setminus i)$. ■

We conclude the paper with some thoughts about extending Theorem 3.1 and Corollary 3.3 to the case of infinitely many alternatives. As to extending Theorem 3.1, which states that a domain is a deterministic extreme point domain if and only if each strategy-proof and unanimous strict probabilistic rule can be written as a convex combination of two other strategy-proof and unanimous probabilistic rules, for the infinite case one may try and find a suitable topology on the set of all such rules so that it becomes a convex and compact subset of a topological vector space. Then, one could apply a topological version of the Krein–Milman Theorem (e.g., Theorem III.4.1 in Barvinok, 2002) and conclude that each strategy-proof and unanimous probabilistic rule is in the closure of the convex hull of the strategy-proof and unanimous deterministic rules. This, however, does not seem a straightforward exercise, and also does not deliver the exact analogue of Theorem 3.1. Next, Corollary 3.3 states that for the case of finitely many alternatives every binary support domain is a deterministic extreme point domain. This is a direct consequence of Theorems 3.1 and 3.2, where the latter theorem states that every strategy-proof and unanimous strict probabilistic rule assigning positive probability to only two alternatives x and y, can be written as a convex combination of two other such rules. Again, extending this theorem to the case of infinitely many alternatives does not seem to be a sinecure: the proof for the finite case heavily uses the fact that if a probability $p \in (0, 1)$ is assigned to x at some preference profile, then we can find an interval around p such that at each other profile either probability p is assigned to x or some probability outside this interval. A proof along this line seems to break down if there are infinitely many alternatives.

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